

A note on a famous theorem of Pang and Zalcman

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Abstract

In this paper, by studying the famous theorem of Pang and Zalcman, we find a normal family and obtain a result, which is an improvement of Pang and Zalcman's theorem in some sense. Meanwhile, several examples are provided to show that our result's conditions are necessary.

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1 Introduction

Let D be a domain in \mathbb{C} , let f be a meromorphic function on D , and let S be a set with the finite elements. Set

$$\overline{E}_f(S) = f^{-1}(\{S\}) \cap D = \{z \in D : f(z) \in S\}.$$

In this paper, we assume that f, g are two meromorphic functions on D and S_1, S_2 are two sets. We denote $\overline{E}_f(S_1) \subset \overline{E}_g(S_2)$ by $f(z) \in S_1 \Rightarrow g(z) \in S_2$. If $\overline{E}_f(S_1) = \overline{E}_g(S_2)$, we denote this condition by $f(z) \in S_1 \Leftrightarrow g(z) \in S_2$. If the set S has only one element, say a , we denote $f(z) \in S$ by $f(z) = a$ (see [16]).

Now, let \mathcal{F} be a family of meromorphic functions on a domain D . We say that \mathcal{F} is normal in D if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains either a subsequence which converges to a meromorphic function f uniformly on each compact subset of D or a subsequence which converges to ∞ uniformly on each compact subset of D (see. [12]).

According to Bloch's principle, a lot of normality criteria have been obtained by starting from Picard type theorems. On the other hand, by Nevanlinna's famous five point theorem and Montel's theorem, it is interesting to establish normality criteria by using conditions known from a uniqueness theorem. A first attempt to this was made by W. Schwick (see. [13]).

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Up to now, many normality criteria have been obtained in this direction.(see. [1, 2, 3, 4, 6, 7, 8, 9, 11, 14, 15]). In 2000, Pang and Zalcman [11] proved a famous theorem.

Theorem A. *Let \mathcal{F} be a family of functions meromorphic on a domain, all of whose zeros are of multiplicity (at least) k . If there exist $b \neq 0$ and $h > 0$ such that for every $f \in \mathcal{F}$, $\overline{E}_f(0) = \overline{E}_{f^{(k)}}(b)$ and $0 < |f^{(k+1)}(z)| \leq h$ whenever $z \in \overline{E}_f(0)$, then \mathcal{F} is normal in D .*

It is natural to ask whether Theorem A still holds if the condition $\overline{E}_f(0) = \overline{E}_{f^{(k)}}(b)$ is replaced by $\overline{E}_f(0) \subset \overline{E}_{f^{(k)}}(b)$. Unfortunately, we neither give a negative example nor prove it true. This problem is very difficult even for the family of holomorphic functions(see. [1, 2, 15]). In this note, we study the special case that $k = 2$ and obtain the following result.

Theorem 1. *Let \mathcal{F} be a family of functions holomorphic on a domain D , all of whose zeros are of multiplicity (at least) 2. If there exist a non-zero constant b and a positive constant M such that for every $f \in \mathcal{F}$,*

- (1) $f(z) = 0 \Rightarrow f''(z) = b$,
- (2) $f''(z) = b \Rightarrow 0 < |f'''(z)| \leq M$ and
- (3) $f'^2(z) = Bf(z)$ whenever $z \in \overline{E}_{f''}(b)$,

where B is a non-constant, then \mathcal{F} is normal in D .

Remark 1. Here, if f omits a constant b , we can say that all the zeros of $f - b$ are of multiplicity ∞ .

Remark 2. For the special cases that \mathcal{F} is holomorphic functions and $k = 2$ of Theorem A, from $\overline{E}_f(0) = \overline{E}_{f''}(b)$, it is easy to deduce \mathcal{F} satisfies the condition (3) of Theorem 1. Thus, in some sense, our result is an improvement of Theorem A. Meanwhile, we know that the condition $\overline{E}_f(0) = \overline{E}_{f^{(k)}}(b)$ is not necessary for holomorphic functions in Theorem A.

Remark 3. We give an example to show that there exists a normal family \mathcal{F} satisfying the conditions of Theorem 1.

Consider the family $\mathcal{F} = \{f_n, n = 1, 2, \dots\}$ on the unit disc, where

$$f_n(z) = e^{\frac{z}{n}},$$

so that

$$f'_n(z) = \frac{1}{n} e^{\frac{z}{n}} \quad \text{and} \quad f''_n(z) = \frac{1}{n^2} e^{\frac{z}{n}}.$$

Let b be a non-zero constant and $B = b$. Then, it is easy to see the family \mathcal{F} satisfies the conditions of Theorem 1 and \mathcal{F} is normal on the unit disc.

Remark 4. The assumption $0 < |f''(z)| \leq M$ cannot be replaced by $|f''(z)| \leq M$. We have a counter-example [11] to show it.

Consider the family $\mathcal{F} = \{f_n, n = 1, 2, \dots\}$ on the unit disc, where

$$f_n(z) = \frac{1}{n^2}(e^{nz} + e^{-nz} - 2) = \frac{1}{n^2}e^{-nz}(e^{nz} - 1)^2,$$

so that

$$f_n^{(j)}(z) = n^{(j-2)}[e^{nz} + (-1)^j e^{-nz}], \quad j = 1, 2, \dots$$

It is easy to see all the zeros of f_n are of multiplicity 2 and

$$f_n(z) = 0 \Leftrightarrow f_n''(z) = 2 \Rightarrow f_n'''(z) = 0.$$

While the family \mathcal{F} is not normal on the unit disc.

2 Some Lemmas

In order to prove our theorems, we need several lemmas. For the convenience of the reader, we recall these lemmas here.

The following result is due to Pang and Zalcman, see [11].

Lemma 1. *Let \mathcal{F} be a family of functions holomorphic on the unit disc, all of whose zeros have multiplicity at least k , and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, if \mathcal{F} is not normal, then there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$;*

(b) *points $z_n, z_n < r$;*

(c) *functions $f_n \in \mathcal{F}$, and*

(d) *positive number $\rho_n \rightarrow 0$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \rightarrow g(\xi)$ locally uniformly, where g is a nonconstant holomorphic function on \mathbb{C} , whose zeros have multiplicity at least k , such that $g^\sharp(\xi) \leq g^\sharp(0) = A + 1$ and $\rho(g) \leq 1$.*

Here, as usual, $g^\sharp(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative and $\rho(g)$ is the order of g .

Next, we need to introduce a result, see [5, Theorem 4.1] or [10], which plays an important part in the proof of our Theorem.

Lemma 2. *Let f be an entire function of order at most 1 and k be a positive integer, then*

$$m(r, \frac{f^{(k)}}{f}) = o(\log r), \quad \text{as } r \rightarrow \infty.$$

Finally, we recall the theorem of Chang, Fang and Zalcman, see [3], which is crucial to the proof of our theorem.

Lemma 3. *Let g be a non-constant entire function with $\rho(g) \leq 1$, let $k \geq 2$ be an integer, and let a be a non-zero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then*

$$g(z) = a(z - z_0),$$

where z_0 is a constant.

3 Proof of Theorem 1

Now, we prove Theorem 1. For every $f \in \mathcal{F}$, it follows from the assumption (1) that all the zeros of f have multiplicity 2. Noting that f is holomorphic in D , we can set

$$f = h^2, \quad (3.1)$$

where h is holomorphic in D . Differentiating (3.1) yields

$$f' = 2hh', \quad f'' = 2(h'^2 + hh'') \quad \text{and} \quad f''' = 6h'h'' + 2hh'''. \quad (3.2)$$

We know that if $\mathcal{H} = \{h\}$ is normal in D , then \mathcal{F} is normal in D . Thus, we need only to prove that \mathcal{H} is normal in D . Suppose, to the contrary, that \mathcal{H} is not normal in D .

It is clear from (3.1), the middle function of (3.2) and the condition (1) that

$$h = 0 \Rightarrow h' \in \{a, -a\} \quad (3.3)$$

where $2a^2 = b$. Combining the condition (2) and the last two functions of (3.2) yields

$$2(h'^2 + hh'') = b \Rightarrow 0 < |6h'h'' + 2hh'''| \leq M.$$

By Lemma 1, we can find $|z_n| < 1$, $\rho_n \rightarrow 0$ and $h_n \in \mathcal{H}$ such that

$$g_n(\xi) = \rho_n^{-1} h_n(z_n + \rho_n \xi) \rightarrow g(\xi) \quad (3.4)$$

locally uniformly on \mathbb{C} , where g is a non-constant entire function such that $g^\sharp(\xi) \leq g^\sharp(0) = M_1 = |a| + 1$. In particular $\rho(g) \leq 1$.

From (3.4), it is easy to obtain that

$$g'_n(\xi) = h'_n(z_n + \rho_n \xi) \rightarrow g'(\xi) \quad (3.5)$$

and

$$g''_n(\xi) = \rho_n h''_n(z_n + \rho_n \xi) \rightarrow g''(\xi)$$

locally uniformly on \mathbb{C} . Let

$$H_n(\xi) = 2[(g'_n(\xi))^2 + g_n(\xi)g''_n(\xi)].$$

Then, a routine calculation leads to

$$H_n(\xi) = f''_n(z_n + \rho_n \xi).$$

Set

$$G = 2(g'^2 + gg''). \quad (3.6)$$

Thus, we can deduce that

$$H_n(\xi) = 2[(g'_n(\xi))^2 + g_n(\xi)g''_n(\xi)] = f''_n(z_n + \rho_n \xi) \rightarrow 2[g'^2(\xi) + g(\xi)g''(\xi)] = G(\xi) \quad (3.7)$$

locally uniformly on \mathbb{C} .

We claim that

$$(I) \quad g(\xi) = 0 \Rightarrow g'(\xi) \in \{a, -a\},$$

$$(II) \quad g(\xi) = 0 \Rightarrow G(\xi) = b \text{ and}$$

$$(III) \quad G(\xi) = b \Rightarrow G'(\xi) = 0.$$

First we prove (I).

Suppose that $g(\xi_0) = 0$, then by Hurwitz's theorem and (3.4), there exist a sequence $\{\xi_n\}$ such that $\xi_n \rightarrow \xi_0$ and (for n sufficiently large)

$$g_n(\xi_n) = \rho_n^{-1} h_n(z_n + \rho_n \xi_n) = 0.$$

Thus $h_n(z_n + \rho_n \xi_n) = 0$. It is clear from (3.3) that

$$h'_n(z_n + \rho_n \xi_n) \in \{a, -a\}.$$

By (3.5), we obtain

$$g'(\xi_0) = \lim_{n \rightarrow \infty} h'_n(z_n + \rho_n \xi_n) \in \{a, -a\},$$

which implies $g(\xi) = 0 \Rightarrow g'(\xi) \in \{a, -a\}$. It is (I).

Similarly as above, we can get (II).

We prove (III) as follows.

We affirm that $G \neq b$. Otherwise, suppose that $G = b$. That is

$$2(g'^2 + gg'') = b.$$

Integrating the above differential equation yields $2gg' = bz + c$, where c is a constant.

If g is a polynomial, then the equation $2gg' = bz + c$ implies that $\deg(g) = 1$. From (I), we get $g' = a$ or $-a$. Then

$$|a| + 1 = g^\sharp(0) \leq |g'(0)| = |a| < |a| + 1,$$

a contradiction.

If g is a transcendental entire function, then g' is also a transcendental entire function. By the lemma of logarithmic derivative, we have

$$\begin{aligned} 2T(r, g') &= T(r, g'^2) = m(r, g'^2) \leq m(r, \frac{g'^2}{gg'}) + m(r, gg') \\ &= m(r, \frac{g'}{g}) + m(r, (bz + c)/2) = S(r, g) = S(r, g'), \end{aligned}$$

which is a contradiction. Thus, we finish the proof of $G \neq b$.

Now, we return to the proof of (III).

Suppose that $G(\zeta_0) = b$. By Hurwitz's theorem and (3.7), there exist a sequence $\{\zeta_n\}$ such that $\zeta_n \rightarrow \zeta_0$ and (for n sufficiently large)

$$H_n(\zeta_n) = f''_n(z_n + \rho_n \zeta_n) = b.$$

It follows from the assumption (2) that

$$0 < |f'''_n(z_n + \rho_n \zeta_n)| \leq M.$$

With (3.7), we deduce

$$H'_n(\xi) = \rho_n f_n'''(z_n + \rho_n \xi) \rightarrow G'(\xi)$$

locally uniformly on \mathbb{C} . Thus, it is not difficult to deduce that

$$G'(\zeta_0) = \lim_{n \rightarrow \infty} \rho_n f_n'''(z_n + \rho_n \zeta_n) = 0,$$

which implies (III).

Now, we continue to prove our theorem.

Suppose that η_0 is a zero of g . That is $g(\eta_0) = 0$. By the claim (I) and (II), we get $g'(\eta_0) = a$ or $-a$ and $G(\eta_0) = b$. Differentiating (3.6) yields that

$$G' = 6g'g'' + 2gg'''. \quad (3.8)$$

It is clear from (III) and (3.8) that

$$G'(\eta_0) = 6g'(\eta_0)g''(\eta_0) + 2g(\eta_0)g'''(\eta_0) = 0.$$

Then, we obtain $g''(\eta_0) = 0$, which implies that

$$g(\xi) = 0 \Rightarrow g''(\xi) = 0.$$

Suppose that g is a polynomial with $\deg g = n$. Noting that (I), we know that g has only simple zeros. Thus, g has n distinct zeros z_m ($m = 1, 2, \dots, n$). By (I), we get $g'(z_m) = a$ or $-a$ ($m = 1, 2, \dots, n$). Thus, either $g' - a$ or $g' + a$ has at least p distinct zeros, here $p = \frac{n}{2}$ if n is an even number, $p = \frac{n+1}{2}$ if n is an odd number. Without loss of generality, we assume that $g'(z_m) - a = 0$ ($m = 1, 2, \dots, p$). Obviously, $g''(z_m) = 0$ ($m = 1, 2, \dots, p$). It implies that each z_m ($m = 1, 2, \dots, p$) is a multiple zero of $g' - a$. Furthermore, it is easy to deduce that

$$n - 1 = \deg(g') = \deg(g' - a) \geq 2p \geq n,$$

a contradiction.

All the foregoing discussion shows that g is a transcendental entire function. Set

$$\phi = \frac{g''}{g}. \quad (3.9)$$

We find that ϕ is an entire function and $\rho(\phi) \leq \rho(g) \leq 1$. Combining Lemma 2 and the lemma of logarithmic derivative yields

$$T(r, \phi) = m(r, \phi) = m(r, \frac{g'}{g}) = o(\log r),$$

which implies ϕ is a non-zero constant. By solving the differential equation (3.9), we have

$$g = c_1 e^{\lambda \xi} + c_2 e^{-\lambda \xi}, \quad (3.10)$$

where c_1, c_2 are two constants and $\lambda^2 = \phi$.

Next, we prove that neither c_1 nor c_2 is zero. Otherwise, without loss of generality, suppose that $c_2 = 0$. Combining (3.6) and (3.10) yields

$$G(\xi) = 4c_1^2\lambda^2e^{2\lambda\xi}$$

and

$$G'(\xi) = 8c_1^2\lambda^3e^{2\lambda\xi}.$$

From (III) and the above two functions, it is easy to deduce a contradiction. Thus, we finish the proof of that c_1, c_2 are two non-zero constants.

Differentiating the function g yields

$$g'(\xi) = \lambda[c_1e^{\lambda\xi} - c_2e^{-\lambda\xi}] \quad (3.11)$$

and

$$g''(\xi) = \lambda^2[c_1e^{\lambda\xi} + c_2e^{-\lambda\xi}]. \quad (3.12)$$

From (3.9), it is obvious that

$$g(\xi) = 0 \Leftrightarrow g''(\xi) = 0. \quad (3.13)$$

By (3.10), we get

$$g(\xi) = 0 \Leftrightarrow e^{\lambda\xi} \in \{A, -A\},$$

here $A = \sqrt{-\frac{c_2}{c_1}}$. From (I), we can see that

$$e^{\lambda\xi} = A \Rightarrow g'(\xi) \in \{a, -a\}.$$

Noting that the form of g' , without loss of generality, we can assume that

$$e^{\lambda\xi} = A \Rightarrow g'(\xi) = a.$$

Thus, we have

$$g'(\xi) - a = e^{-\lambda\xi}[c_1\lambda e^{2\lambda\xi} - ae^{\lambda\xi} - c_2\lambda] = A_1e^{-\lambda\xi}[e^{\lambda\xi} - A][e^{\lambda\xi} - A_2], \quad (3.14)$$

where A_1 and A_2 are two non-zero constants. Observing that (3.13), we get

$$e^{\lambda\xi} = A \Rightarrow g''(\xi) = 0,$$

which implies that all the zeros of $e^{\lambda\xi} - A$ are multiple zeros of $g' - a$. Therefore, we deduce that $A_2 = A$. Rewriting (3.14) as

$$g'(\xi) - a = A_1e^{-\lambda\xi}[e^{\lambda\xi} - A]^2.$$

It indicates that $g'(\xi) = a \Leftrightarrow e^{\lambda\xi} = A$. Meanwhile, with the same argument, we can deduce that $g'(\xi) = -a \Leftrightarrow e^{\lambda\xi} = -A$. Combining the two cases yields that $g'(\xi) \in \{a, -a\} \Leftrightarrow e^{\lambda\xi} \in \{A, -A\}$. Thus, we have

$$g(\xi) = 0 \Leftrightarrow g'(\xi) \in \{a, -a\}.$$

Furthermore, we obtain

$$g = 0 \Leftrightarrow g' \in \{a, -a\} \Leftrightarrow g'' = 0 \Rightarrow G = b. \quad (3.15)$$

Noting that (3.11), we know $g' - a$ has multiple zeros. Differentiating (3.12) yields

$$g''' = \lambda^3 [c_1 e^{\lambda \xi} - c_2 e^{-\lambda \xi}].$$

From the above function, it is not difficult to deduce that $g' - a$ has zeros with multiplicity 2.

Suppose $g'(\alpha_0) = a$. By (3.15) we get $g(\alpha_0) = 0$ and $G(\alpha_0) = b$. From (III), we find that α_0 is a multiple zero of $G - b$. Noting that $G \neq b$, then there exists $\delta > 0$ such that

$$g(\xi) \neq 0, \quad G(\xi) - b \neq 0,$$

in $D'(\alpha_0, \delta) = \{\xi : 0 < |\xi - \alpha_0| < \delta\}$. By (3.7), there exists $\varepsilon_0 > 0$ such that, for each $0 < \delta' < \delta$ and sufficiently large n ,

$$|f_n''(z_n + \rho_n \xi) - b - (G(\xi) - b)| < \varepsilon_0 < |G(\xi) - b|$$

on the circle $C(\alpha_0, \delta') = \{\xi : |\xi - \alpha_0| = \delta'\}$. By Rouché theorem, there exist $\{\alpha_{n,j}\}$ ($j = 1, 2$) tending to α_0 , such that, for each large n

$$H_n(\alpha_{n,j}) = f_n''(z_n + \rho_n \alpha_{n,j}) = b \quad (j = 1, 2). \quad (3.16)$$

And the assumption (2) implies that $\alpha_{n,1} \neq \alpha_{n,2}$. Then, for $j = 1, 2$, it follows from the assumption (3) that

$$f_n'(z_n + \rho_n \alpha_{n,j})^2 = B f_n(z_n + \rho_n \alpha_{n,j}). \quad (3.17)$$

We distinguish the following three cases.

Case 1. For $j = 1, 2$, there exist infinitely many n_t satisfying

$$f_{n_t}(z_{n_t} + \rho_{n_t} \alpha_{n_t,j}) = 0.$$

Then we get $h_{n_t}(z_{n_t} + \rho_{n_t} \alpha_{n_t,j}) = 0$ ($j = 1, 2$). It follows from (3.4) and Rouché theorem that α_0 is a zero of g with multiplicity at least 2. But g has only simple zeros, a contradiction.

Case 2. For $j = 1, 2$, there exist infinitely many n_t satisfying

$$f_{n_t}(z_{n_t} + \rho_{n_t} \alpha_{n_t,j}) \neq 0.$$

We claim that there exists a subsequence of $\{n_t\}$ (we still denote it by $\{n_t\}$) which contains infinite elements satisfying

$$h'_{n_t}(z_{n_t} + \rho_{n_t} \alpha_{n_t,j}) = a \quad (j = 1, 2). \quad (3.18)$$

Without loss of generality, we need only to prove it holds for $j = 1$. By (3.1), the first item of (3.2) and (3.17), it is not difficult to deduce

$$h'_{n_t}(z_{n_t} + \rho_{n_t} \alpha_{n_t,1}) \in \{d, -d\},$$

where $d = \frac{\sqrt{B}}{2}$ is a constant. It is clear from the assumption $f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) \neq 0$ that d is a non-zero constant.

Then, there must exists a subsequence of $\{n_t\}$ (we still denote it by $\{n_t\}$) which contains infinite elements satisfying

$$h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = e, \quad (3.19)$$

here $e \in \{d, -d\}$ is a non-zero constant. Then

$$g'(\alpha_0) = \lim_{n \rightarrow \infty} h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = e.$$

Noting that $g'(\alpha_0) = a$, we get $e = a$. With (3.19), we prove the claim.

On the other hand, by the middle item of (3.2), (3.16), (3.18) and the assumption of Case 2, we can deduce $h''_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) = 0$ for $j = 1, 2$.

Observing that $h''_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,j}) = 0$ for $j = 1, 2$, so each $\alpha_{n_t,j}$ ($j = 1, 2$) is a multiple zero of $h'_{n_t}(z_{n_t} + \rho_{n_t}\xi) - a$. It follows from (3.5) and Rouché theorem that α_0 is a zero of $g' - a$ with multiplicity at least 4, a contradiction.

Case 3. There exist infinitely many n_t satisfying either

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = 0, \quad f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) \neq 0$$

or

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) \neq 0, \quad f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) = 0.$$

Without loss of generality, suppose that

$$f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = 0 \quad \text{and} \quad f_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) \neq 0.$$

Similarly as Case 2, there exists a subsequence of $\{n_t\}$ (we still denote it by $\{n_t\}$) which contains infinite elements satisfying

$$h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,1}) = a,$$

$$h'_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) = a \quad \text{and} \quad h''_{n_t}(z_{n_t} + \rho_{n_t}\alpha_{n_t,2}) = 0.$$

That means $\alpha_{n_t,2}$ is a multiple zero of $h'_{n_t}(z_{n_t} + \rho_{n_t}\xi) - a$. Meanwhile, $h'_{n_t}(z_{n_t} + \rho_{n_t}\xi) - a$ has another zero $\alpha_{n_t,1}$. Then, it follows from (3.6) and Rouché theorem that α_0 is a zero of $g' - a$ with multiplicity at least 3, a contradiction.

Thus, we get $g'(\alpha_0) \neq a$, which is a contradiction.

All the above discussion yields \mathcal{H} is normal in D , so \mathcal{F} is also normal in D .

Hence, we complete the proof of Theorem 1.

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